SUGGESTED SOLUTION TO TEST

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Problem 1. Let X be a normed space.

(i) State the Uniform Boundedness Theorem.

(ii) Show that if $\{x_n\}_{n\geq 1}$ is a weakly convergent sequence in X, then its limit is unique and $\{x_n\}_{n\geq 1}$ is bounded.

Proof. (i) The uniform boundedness theorem states that: If $\mathfrak{F} = \{T_j\}_{j\in J}$ is a family of bounded operators from a Banach space X to a normed space Y such that $\sup\limits_{j\in J}\|T_j x\|_Y < \infty$ for every $x \in X$, then $\sup\limits_{j\in J}\|T_j\|_X < \infty$.

(ii) To prove the uniqueness, suppose $\{x_n\}_{n\geq 1}$ weakly converges to $x, x' \in X$, then for all $f \in X^*$, for arbitrary $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$|f(x_n) - f(x)| < \frac{\varepsilon}{2}, \quad |f(x_n) - f(x')| < \frac{\varepsilon}{2},$$

therefore

$$|f(x - x')| = |f(x) - f(x')| \leq |f(x) - f(x_n)| + |f(x_n) - f(x')| < \varepsilon,$$

which implies that $f(x - x') = 0$ by the arbitrariness of ε, then by Hahn-Banach theorem, we have $x = x'$.

To prove the boundedness, for $x \in X$, we define $\hat{x} \in X^{**}$ by

$$\hat{x}(f) := f(x),$$

for all $f \in X^*$, then $\hat{x}_n \in X^{**}$, moreover, by Hahn-Banach theorem, $\|\hat{x}\|_{X^{**}} = \|x\|_X$. Since $\{x_n\}_{n\geq 1}$ weakly converges to x, therefore for all $f \in X^*$, $\{|\hat{x}_n(f)|\}_{n\geq 1}$ is bounded, by the uniform boundedness theorem, we have $\{\|\hat{x}_n\|_{X^{**}}\}_{n\geq 1}$ is bounded, which implies that $\{\|x_n\|_X\}_{n\geq 1}$ is bounded. □

Problem 2. Let X and Y be the normed spaces.

(i) Show that if Y is of finite dimension, then a linear map $T : X \to Y$ is bounded if and only if $\ker T$ is closed.

(ii) Show that every finite dimensional subspace M of X is complemented, that is, there is a closed subspace N of X so that $X = M \oplus N$.

Proof. (i) $\Rightarrow$: Suppose the linear map $T : X \to Y$ is bounded, then T is also continuous, therefore $\ker T$ is closed.

$\Leftarrow$: Suppose $\ker T$ is closed, then $X/\ker T$ with quotient norm is a normed space. We define the map $\overline{T} : X/\ker T \to Y$ by

$$\overline{T}(\overline{x}) := T(x),$$

for all $\overline{x} = x + \ker T \in X/\ker T$. It is clear that $\overline{T}$ is well-defined. Since $X/\ker T$ is isomorphic to a subspace of Y, therefore $X/\ker T$ is of finite dimension, then

$\overline{T}$ is continous. Let $\pi : X \to X/\mathrm{ker}T$ denote the quotient map, since π is also continuous, then by the composition,

$$T = \overline{T} \circ \pi,$$

T is continuous.

(ii) Assume M is of dimension $m \in \mathbb{N}$. Let $\{e_i\}_{1\le i\le m}$ be the basis of M, we define the linear functional on M by

$$c_i(e_j) := \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

By Hahn-Banach theorem, there exists $\tilde{c}_i \in X^*$ such that

$$\tilde{c}_i|_M = c_i, \quad |\tilde{c}_i(x)| \le \|x\|_X.$$

Let $N = \bigcap_{i=1}^{m} \mathrm{ker}\tilde{c}_i$, we claim that $X = M \oplus N$. It is clear that for arbitrary $x \in X$, we have

$$x = \sum_{i=1}^{m} \tilde{c}_i(x)e_i + \left(x - \sum_{i=1}^{m} \tilde{c}_i(x)e_i \right) := x_M + x_N.$$

It is clear that $x_M \in M$ and $x_N \in N$. Suppose $x \in M \cap N$, then on the one hand, there exist constants $\lambda_1, \cdots, \lambda_m$ such that

$$x = \sum_{i=1}^{m} \lambda_i e_i,$$

on the other hand,

$$\tilde{c}_i(x) = 0, \quad i = 1, \cdots, m,$$

therefore

$$\lambda_i = 0, \quad i = 1, \cdots, m,$$

which implies that $x = 0$. □

Problem 3. Let V be a vector space over $\mathbb{R}$. Let $\rho : V \to [0, \infty)$ be a function on V satisfying the condition: $\rho(tx) = t\rho(x)$ for all $x \in V$ and for all $t \ge 0$. Show that a linear functional $f : V \to \mathbb{R}$ is ρ-continuous on V if and only if there is $C > 0$ such that $|f(x)| \le C\rho(x)$ for all $x \in V$. (In here, ρ-continuous means that for all $x_0 \in V$ and for all $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $\rho(x - x_0) < \delta$.)

Proof. $\Rightarrow$: Suppose the linear functional $f : V \to \mathbb{R}$ is ρ-continuous, then f is ρ-continuous at $x = 0$, which implies that there exists $\delta > 0$ such that for all $y \in V$ such that $\rho(y) < \delta$,

$$|f(y)| < 1.$$

Let $x \in V$, we have

$$\rho\left(\frac{\delta}{2\rho(x)}x\right) = \frac{\delta}{2} < \delta,$$

therefore

$$\left|f\left(\frac{\delta}{2\rho(x)}x\right)\right| < 1,$$

which implies that

$$|f(x)| < \frac{2}{\delta}\rho(x).$$

$\Leftarrow$: Suppose for all $x \in V$,

$$|f(x)| \leq C\rho(x).$$

Then for arbitrary $\varepsilon > 0$, and all $y \in V$ such that $\rho(y - x) < \frac{\varepsilon}{C}$, we have

$$|f(x) - f(x)| \leq C\rho(x - y) < \varepsilon,$$

which implies that f is ρ-continuous. □

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